

# An Elimination Lemma for Algebras with PBW Bases

Huishi Li\*

Department of Applied Mathematics, College of Information Science and Technology  
Hainan University, Haikou 570228, China

**Abstract.** Let  $K$  be a field, and  $A = K[a_1, \dots, a_n]$  a finitely generated  $K$ -algebra with the PBW  $K$ -basis  $\mathcal{B} = \{a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$ . It is shown that if  $L$  is a nonzero left ideal of  $A$  with  $\text{GK.dim}(A/L) = d < n$  (= the number of generators of  $A$ ), then  $L$  has the *elimination property* in the sense that  $\mathbf{V}(U) \cap L \neq \{0\}$  for every subset  $U = \{a_{i_1}, \dots, a_{i_{d+1}}\} \subset \{a_1, \dots, a_n\}$  with  $i_1 < i_2 < \dots < i_{d+1}$ , where  $\mathbf{V}(U) = K\text{-span}\{a_{i_1}^{\alpha_1} \cdots a_{i_{d+1}}^{\alpha_{d+1}} \mid (\alpha_1, \dots, \alpha_{d+1}) \in \mathbb{N}^{d+1}\}$ . In terms of the structural properties of  $A$ , it is also explored when the condition  $\text{GK.dim}(A/L) < n$  may hold for a left ideal  $L$  of  $A$ . Moreover, from the viewpoint of realizing the elimination property by means of Gröbner bases, it is demonstrated that if  $A$  is in the class of binomial skew polynomial rings [G-I2, Serdica Math. J., 30(2004)] or in the class of solvable polynomial algebras [K-RW, J. Symbolic Comput., 9(1990)], then every nonzero left ideal  $L$  of  $A$  satisfies  $\text{GK.dim}(A/L) < \text{GK.dim} A = n$  (= the number of generators of  $A$ ), thereby  $L$  has the elimination property.

**MSC 2010** Primary 13P10; Secondary 16W70, 68W30 (16Z05).

**Key words** Elimination, PBW  $K$ -basis, Filtration, Gelfand-Kirillov dimension, Gröbner basis.

## 0. Introduction

Let  $R = K[x_1, \dots, x_n]$  be the commutative polynomial  $K$ -algebra in  $n$  variables over a field  $K$ , and let  $\{x_{i_1}, \dots, x_{i_s}\} \subset \{x_1, \dots, x_n\}$  with  $i_1 < i_2 < \dots < i_s$ . We

---

\*e-mail: huishipp@yahoo.com

say that a monomial ordering on  $R$  is an *elimination ordering of type  $s$*  with respect to the subalgebra  $K[x_{i_1}, \dots, x_{i_s}]$ , denoted  $\prec_s$ , if  $f \in R$  with the leading monomial  $\mathbf{LM}_{\prec_s}(f) \in K[x_{i_1}, \dots, x_{i_s}]$  implies  $f \in K[x_{i_1}, \dots, x_{i_s}]$ . Let  $I$  be a nonzero ideal of  $R$ . Then it follows from Buchberger's Gröbner basis theory that there is the *Elimination Theorem for ideals of  $R$* , which states that

- If  $\mathcal{G}$  is a Gröbner basis of  $I$  with respect to  $\prec_s$ , then  $\mathcal{G}_s = \mathcal{G} \cap K[x_{i_1}, \dots, x_{i_s}]$  is a Gröbner basis of the ideal  $I \cap K[x_{i_1}, \dots, x_{i_s}]$  in  $K[x_{i_1}, \dots, x_{i_s}]$ .

Obviously, from this theorem we may see that  $\mathcal{G}_s = \emptyset \Leftrightarrow I \cap K[x_{i_1}, \dots, x_s] = \{0\}$ . So, without involving Gröbner basis theory, for an arbitrarily given proper ideal  $I$ , it is natural to ask

- To what extent can the elimination of certain variables happen in  $I$  via pure structural properties of an ideal?

As the literature shows, so far perhaps the best answer to the above question is the one coming from the dimension theory in commutative algebraic geometry. More precisely, recall from [Grö, 1968, 1970] that a subset  $U = \{x_{i_1}, \dots, x_{i_r}\} \subset \{x_1, \dots, x_n\}$  with  $i_1 < i_2 < \dots < i_r$  is said to be *independent (mod  $I$ )* if  $I \cap K[x_{i_1}, \dots, x_{i_r}] = \{0\}$ ; otherwise  $U$  is called *dependent (mod  $I$ )*. Considering the dimension  $\dim \mathcal{V}(I)$  of the affine algebraic set  $\mathcal{V}(I)$ , it is now well known from the literature (e.g. [KW], [BW]) that

$$\begin{aligned} \dim \mathcal{V}(I) &= \max \left\{ |U| \mid U \subset \{x_1, \dots, x_n\} \text{ independent (mod } I) \right\} \\ &= \text{degree of the Hilbert polynomial of } R/I. \end{aligned}$$

Clearly, the above result tells us that

- if  $\dim \mathcal{V}(I) = d < n$ , then  $K[x_{i_1}, \dots, x_{i_{d+1}}] \cap I \neq \{0\}$  for every subset  $U_{d+1} = \{x_{i_1}, \dots, x_{i_{d+1}}\} \subset \{x_1, \dots, x_n\}$  with  $i_1 < i_2 < \dots < i_{d+1}$ , i.e., there are nonzero elements of  $I$  that only depend on the generators in  $U_{d+1}$ , in particular,  $K[x_1, \dots, x_d, x_{d+i}] \cap I \neq \{0\}$ ,  $i = 1, \dots, n - d$ .

At this stage, if  $\dim \mathcal{V}(I) = d < n$  and we want to take out a nonzero polynomial from  $K[x_{i_1}, \dots, x_{i_{d+1}}] \cap I$ , then, with an elimination ordering  $\prec_{d+1}$  of type  $d + 1$  with respect to  $K[x_{i_1}, \dots, x_{i_{d+1}}]$ , running Buchberger's algorithm will produce a Gröbner basis  $\mathcal{G}$  for  $I$  such that  $\mathcal{G}$  contains a nonzero polynomial of  $K[x_{i_1}, \dots, x_{i_{d+1}}]$ .

Here let us point out that in algorithmically computing  $\dim \mathcal{V}(I)$  by using a Gröbner basis of  $I$  ([KW], [BW]), a *strong independence* of  $U$  (mod  $I$ ) for a subset  $U \subset \{x_1, \dots, x_n\}$  was introduced to make the *key* link between the independence of  $U$  (mod  $I$ ) and a Gröbner basis of  $I$ , and a *graded monomial ordering that respects the total degree of polynomials* was necessarily employed throughout the whole implementing process.

Turning to the noncommutative case, let  $A_n(\mathbb{C}) = \mathbb{C}[x_1, \dots, x_n, \partial_1, \dots, \partial_n]$  be the  $n$ -th Weyl algebra over the field  $\mathbb{C}$  of complex numbers, where  $x_1, \dots, x_n$  are indeterminate over  $\mathbb{C}$ ,  $\partial_i = \frac{\partial}{\partial x_i}$ ,  $1 \leq i \leq n$ , and let  $L$  be a left ideal of  $A_n(\mathbb{C})$ . Then the well-known *elimination lemma for Weyl algebras* [Zei1, Lemma 4.1] states that

- If  $A_n(\mathbb{C})/L$  is a holonomic  $A_n(\mathbb{C})$ -module (i.e. the Gelfand-Kirillov dimension  $\text{GK.dim} A_n(\mathbb{C})/L = n$ ), then, for every  $n+1$  generators out of the  $2n$  generators  $\{x_1, \dots, x_n, \partial_1, \dots, \partial_n\}$  of  $A_n(\mathbb{C})$  there is a nonzero member of  $L$  that only depends on these  $n+1$  generators. In particular, for every  $i = 1, \dots, n$ ,  $L$  contains a nonzero element of the subalgebra  $\mathbb{C}[x_1, \dots, x_n, \partial_i] \subset A_n(\mathbb{C})$ .

As  $A_n(\mathbb{C})$  is a solvable polynomial algebra in the sense of [KR-W] and it admits the pure lexicographic ordering  $x_1 \prec_{\text{lex}} \dots \prec_{\text{lex}} x_n \prec_{\text{lex}} \partial_1 \prec_{\text{lex}} \dots \prec_{\text{lex}} \partial_n$  which is certainly an elimination ordering of type  $n+1$  with respect to the subalgebra  $\mathbb{C}[x_1, \dots, x_n, \partial_{i_1}]$ , i.e.,  $f \in A_n(\mathbb{C})$  with the leading monomial  $\mathbf{LM}(f) \in \mathbb{C}[x_1, \dots, x_n, \partial_{i_1}]$  implies  $f \in \mathbb{C}[x_1, \dots, x_n, \partial_{i_1}]$ , it follows that if  $\text{GK.dim} A_n(\mathbb{C})/L = n$ , then running a noncommutative version of Buchberger's algorithm constructed in [K-RW] will produce a left Gröbner basis  $\mathcal{G}$  for  $L$  such that  $\mathcal{G}$  contains a nonzero element of the subalgebra  $\mathbb{C}[x_1, \dots, x_n, \partial_{i_1}]$ . While concerning the determination of holonomicity of  $A_n(\mathbb{C})/L$  (i.e.  $\text{GK.dim} A_n(\mathbb{C})/L = n$ ) by using Gröbner bases, that may refer to a much more general story about computation of Gelfand-Kirillov dimension for modules over quadric solvable polynomial algebras [Li1, CH.V] (we shall soon come to this point below). Nowadays, the elimination lemma for Weyl algebras [Zei1, Lemma 4.1] has been referred to as the “fundamental lemma” in the automatic proving of holonomic function identities [WZ]. Based on this lemma, effective automatic proving of holonomic function identities has been carried out, and a large class of special function identities has been identified ([PWZ], [Zei2], [Ch], [CS]).

Furthermore, consider a (noncommutative) quadric solvable polynomial algebra  $A = K[a_1, \dots, a_n]$  in the sense of [Li1, CH.III, Section 2], that admits a *graded monomial ordering*  $\prec_{gr}$  respecting every  $a_i$  being of degree 1 (Weyl algebras are typical examples of such algebras). Since  $A$  has the PBW  $K$ -basis  $\mathcal{B} = \{a_1^{\alpha_1} \dots a_n^{\alpha_n} \mid (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$  (see next section for an interpretation of this notion), for a subset  $U = \{a_{i_1}, \dots, a_{i_r}\} \subset \{a_1, \dots, a_n\}$  with  $i_1 < i_2 < \dots < i_r$ ,  $U$  is said to be *weakly independent modulo a left ideal*  $L$  of  $A$  if  $L \cap \mathbf{V}(U) = \{0\}$ , where

$$\mathbf{V}(U) = K\text{-span} \{ a_{i_1}^{\alpha_1} \dots a_{i_r}^{\alpha_r} \mid (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r \}.$$

With this weak independence of  $U \pmod{L}$  and a double filtered-graded transfer trick, the strategy of computing  $\dim \mathcal{V}(I)$  proposed by ([KW], [BW]) was adapted in [Li1, CH.V] for computing the Gelfand-Kirillov dimension  $\text{GK.dim}(A/L)$ , and consequently the following

results were established:

- [Li1, CH.V, Theorem 7.4] Let  $L$  be a *nonzero* left ideal of  $A$ . Then

$$\begin{aligned} \text{GK.dim}(A/L) &= \text{degree of the Hilbert polynomial of } A/L \\ &= \max \left\{ |U| \mid U \subset \{a_1, \dots, a_n\} \text{ weakly independent (mod } L) \right\}, \end{aligned}$$

which can be algorithmically computed via a Gröbner basis of  $L$ ; moreover,

$$\text{GK.dim}(A/L) < n = \text{GK.dim}A;$$

- [Li1, CH.V, Lemma 7.5] If  $\text{GK.dim}A/L = d$ , then  $\mathbf{V}(U) \cap L \neq \{0\}$  for every subset  $U = \{a_{i_1}, \dots, a_{i_{d+1}}\} \subset \{a_1, \dots, a_n\}$  with  $i_1 < i_2 < \dots < i_{d+1}$ .

Therefore, if  $\text{GK.dim}A/L = d$ ,  $U = \{a_{i_1}, \dots, a_{i_{d+1}}\}$  with  $i_1 < i_2 < \dots < i_{d+1}$ , and if  $A$  admits an elimination ordering  $\prec_{d+1}$  of type  $d+1$  with respect to  $\mathbf{V}(U)$ , i.e.,  $f \in A$  with the leading monomial  $\mathbf{LM}(f) \in \mathbf{V}(U)$  implies  $f \in \mathbf{V}(U)$ , then running the noncommutative Buchberger's algorithm constructed [K-RW] will produce a left Gröbner basis  $\mathcal{G}$  for  $L$  such that  $\mathcal{G}$  contains a nonzero element of  $\mathbf{V}(U)$ .

Note that the class of quadric solvable polynomial algebras studied in [Li1, CH.III, CH.V] covers not only Weyl algebras, but also more (skew) Ore extensions and operator algebras. Enlightened by the automatic proving of multivariate identities over operator algebras ([PWZ], [Ch], [CS]), more general  $\partial$ -finiteness and  $\partial$ -holonomicity for modules over quadric solvable polynomial algebras were introduced and preliminarily studied in [Li1, CH.VII] by using [Li1, CH.V, Lemma 7.5] as a key role.

Inspired by the elimination lemma [Zei1, Lemma 4.1] and the elimination Lemma [Li1, CH.V, Lemma 7.5], in this paper we first show that there is a kind of elimination lemma (Elimination Lemma 2.1) for *all* finitely generated  $K$ -algebras with PBW  $K$ -bases, that is, if  $A = K[a_1, \dots, a_n]$  is a finitely generated  $K$ -algebra with the PBW  $K$ -basis  $\mathcal{B} = \{a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$ , and if  $L$  is a nonzero left ideal of  $A$  with  $\text{GK.dim}(A/L) = d < n$  ( $=$  the number of generators of  $A$ ), then  $L$  has the *elimination property* in the sense that  $\mathbf{V}(U) \cap L \neq \{0\}$  for every subset  $U = \{a_{i_1}, \dots, a_{i_{d+1}}\} \subset \{a_1, \dots, a_n\}$  with  $i_1 < i_2 < \dots < i_{d+1}$ , where  $\mathbf{V}(U) = K\text{-span}\{a_{i_1}^{\alpha_1} \cdots a_{i_{d+1}}^{\alpha_{d+1}} \mid (\alpha_1, \dots, \alpha_{d+1}) \in \mathbb{N}^{d+1}\}$ ; then we explore, in terms of the structural properties of  $A$ , when the condition  $\text{GK.dim}(A/L) < n$  may hold for a nonzero left ideal  $L$  of  $A$  (Theorem 2.6). From the viewpoint of realizing the elimination property by means of Gröbner bases, in the last two sections, we demonstrate that if  $A = K[a_1, \dots, a_n]$  is an algebra in either the class of binomial skew polynomial rings [G-I2] or the class of solvable polynomial algebras [K-RW], then every nonzero left ideal  $L$  of  $A$  satisfies  $\text{GK.dim}(A/L) < \text{GK.dim}A = n$ , thereby  $L$  has the elimination property (Theorem 3.3, Theorem 4.1).

Throughout the following sections,  $K$  denotes a field,  $K^* = K - \{0\}$ ;  $\mathbb{N}$  denotes the set of all nonnegative integers, and  $\mathbb{Z}$  denotes the set of all integers; algebras are meant associative  $K$ -algebra with multiplicative identity 1; if  $A = K[a_1, \dots, a_n]$  is a finitely generated  $K$ -algebra, then we always assume that the set of generators  $\{a_1, \dots, a_n\}$  is *minimal*, i.e., any proper subset of  $\{a_1, \dots, a_n\}$  cannot generate  $A$  as a  $K$ -algebra.

## 1. Preliminaries

To reach our goal of this paper, in this section we recall several necessary notions concerning finitely generated  $K$ -algebras and their modules, such as PBW  $K$ -basis,  $\mathbb{N}$ -filtration, and Gelfand-Kirillov dimension; moreover, some known results related to these notions, which will be used in later sections, are recalled as well. A general Gröbner basis theory for ideals of free algebras is referred to [Gr] and [Mor], and a general Gelfand-Kirillov dimension theory for algebras and modules is referred to [KL] and [MR].

Let  $A = K[a_1, \dots, a_n]$  be a finitely generated  $K$ -algebra with the set of generators  $\{a_1, \dots, a_n\}$ . If, for some permutation  $i_1 i_2 \dots i_n$  of  $1, 2, \dots, n$ , the set  $\mathcal{B} = \{a^\alpha = a_{i_1}^{\alpha_1} \dots a_{i_n}^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$ , forms a  $K$ -basis of  $A$ , then  $\mathcal{B}$  is referred to as a *PBW  $K$ -basis* of  $A$  (where the phrase “PBW  $K$ -basis” is abbreviated from the well-known *Poincaré-Birkhoff-Witt Theorem* concerning the standard  $K$ -basis of the enveloping algebra of a Lie algebra, e.g. see [Hu, P. 92]). For more content concerning PBW  $K$ -bases and related topics, the reader is referred to a nice survey paper [SW].

It is clear that if  $A$  has a PBW  $K$ -basis, then we can always assume that  $i_1 = 1, \dots, i_n = n$ . Thus, we make the following convention once for all.

**Convention** From now on in this paper, if we say that a finitely generated algebra  $A = K[a_1, \dots, a_n]$  has the PBW  $K$ -basis  $\mathcal{B}$ , then it always means that

$$\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} \dots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}.$$

Moreover, adopting the commonly used terminology in computational algebra, elements of  $\mathcal{B}$  are referred to as *monomials* of  $A$ .

Let  $K\langle X \rangle = K\langle X_1, \dots, X_n \rangle$  be the free  $K$ -algebra generated by  $X = \{X_1, \dots, X_n\}$ . Then  $K\langle X \rangle$  has the standard  $K$ -basis  $\mathbb{B}$  consisting of words on alphabet  $X$ , or more precisely, writing 1 for the empty word,

$$\mathbb{B} = \{1\} \cup \{X_{i_1} \dots X_{i_s} \mid X_{i_j} \in X, s \geq 1\}.$$

Since every finitely generated  $K$ -algebra  $A = K[a_1, \dots, a_n]$  has a presentation  $K\langle X \rangle / I$  with respect to  $X_i \mapsto a_i$ ,  $1 \leq i \leq n$ , where  $I$  is an ideal of  $K\langle X \rangle$ , the proposition stated below, which is a generalization of [Gr, Proposition 2,14] and [Li1, CH.III, Theorem 1.5], may be viewed as an algorithmic criterion for  $A$  to have the PBW  $K$ -basis  $\mathcal{B}$ .

**1.1. Proposition** [Li4, Ch 4, Theorem 3.1] Let  $I \neq \{0\}$  be an ideal of the free  $K$ -algebra  $K\langle X \rangle = K\langle X_1, \dots, X_n \rangle$ , and  $A = K\langle X \rangle / I$ . Suppose that  $I$  contains a subset of  $\frac{n(n-1)}{2}$  elements

$$G = \{g_{ji} = X_j X_i - F_{ji} \mid F_{ji} \in K\langle X \rangle, 1 \leq i < j \leq n\}$$

such that with respect to some monomial ordering  $\prec_x$  on  $\mathbb{B}$ , the leading monomial  $\mathbf{LM}(g_{ji}) = X_j X_i$  for all  $g_{ji} \in G$ . The following two statements are equivalent:

- (i)  $A$  has the PBW  $K$ -basis  $\mathcal{B} = \{\overline{X_1^{\alpha_1}} \overline{X_2^{\alpha_2}} \cdots \overline{X_n^{\alpha_n}} \mid \alpha_j \in \mathbb{N}\}$  where each  $\overline{X_i}$  denotes the coset of  $I$  represented by  $X_i$  in  $A$ .
- (ii) Any subset  $\mathcal{G}$  of  $I$  containing  $G$  is a Gröbner basis for  $I$  with respect to  $\prec_x$ .  $\square$

**Remark** Obviously, Proposition 1.1 holds true if we use any permutation  $X_{k_1}, X_{k_2}, \dots, X_{k_n}$  of the generators  $X_1, X_2, \dots, X_n$  of  $K\langle X \rangle$  (see an example given in Section 3). So, in what follows we conventionally keep using the natural permutation  $X_1, \dots, X_n$ .

Note that the multiplication of elements in the  $K$ -basis  $\mathbb{B}$  of  $K\langle X \rangle$  is given by the concatenation of words, i.e.,

$$(X_{i_1} \cdots X_{i_s}) \cdot (X_{j_1} \cdots X_{j_t}) = X_{i_1} \cdots X_{i_s} X_{j_1} \cdots X_{j_t}.$$

It follows that  $K\langle X \rangle$  is turned into an  $\mathbb{N}$ -filtered algebra by the natural filtration  $FK\langle X \rangle = \{F_m K\langle X \rangle\}_{m \in \mathbb{N}}$  with

$$F_m K\langle X \rangle = K\text{-span}\{1, X_{i_1} \cdots X_{i_s} \in \mathcal{B} \mid s \leq m\}, \quad m \in \mathbb{N},$$

which satisfies that  $F_0 K\langle X \rangle = K$ , each  $F_m K\langle X \rangle$  is a finite dimensional subspace of  $K\langle X \rangle$ ,  $K\langle X \rangle = \cup_{m \in \mathbb{N}} F_m K\langle X \rangle$ ,  $F_m K\langle X \rangle \subseteq F_{m+1} K\langle X \rangle$ , and  $F_{m_1} K\langle X \rangle F_{m_2} K\langle X \rangle \subseteq F_{m_1+m_2} K\langle X \rangle$ . If  $A = K[a_1, \dots, a_n]$  is a finitely generated  $K$ -algebra, then there is an ideal  $I$  of  $K\langle X \rangle$  and an algebra isomorphism  $K\langle X \rangle / I \xrightarrow{\cong} A$  with  $\overline{X_i} \mapsto a_i$ ,  $1 \leq i \leq n$ , where each  $\overline{X_i} = X_i + I$  is the coset represented by  $X_i$  in  $K\langle X \rangle / I$ . Consequently, with respect to  $\overline{X_i} \mapsto a_i$ ,  $1 \leq i \leq n$ ,  $A$  is turned into an  $\mathbb{N}$ -filtered algebra by the filtration  $FA = \{F_m A\}_{m \in \mathbb{N}}$  induced by  $FK\langle X \rangle$ , i.e., for every  $m \in \mathbb{N}$ ,

$$\begin{aligned} F_m A &= K\text{-span}\{1, a_{i_1} \cdots a_{i_s} \mid a_{i_j} \in \{a_1, \dots, a_n\}, 1 \leq s \leq m\}, \\ &\cong (F_m K\langle X \rangle + I) / I, \end{aligned}$$

which satisfies that  $F_0A = K$ , each  $F_mA$  is a finite dimensional subspace of  $A$ ,  $A = \bigcup_{m \in \mathbb{N}} F_mA$ ,  $F_mA \subseteq F_{m+1}A$ ,  $F_{m_1}AF_{m_2}A \subseteq F_{m_1+m_2}A$  for all  $m_1, m_2 \in \mathbb{N}$  (note that this is determined by the concatenation of “words”  $(a_{i_1} \cdots a_{i_s}) \cdot (a_{k_1} \cdots a_{k_t}) = a_{i_1} \cdots a_{i_s} a_{k_1} \cdots a_{k_t}$ ). Indeed, a direct verification shows that if we take  $V = \sum_{i=1}^n Ka_i$ , then  $F_mA = K + V + V^2 + \cdots + V^m$  for all  $m \in \mathbb{N}$ . Hence, in the literature (cf. [MR, P.26]), the  $\mathbb{N}$ -filtration  $FA$  as described above is usually referred to as the *standard filtration* of  $A$  determined by the *finite dimensional generating subspace*  $V$ . Considering the function  $d_F(m) = \dim_K F_mA$ ,  $m \in \mathbb{N}$ , the Gelfand-Kirillov dimension (abbreviated GK dimension) of  $A$ , denoted  $\text{GK.dim}A$ , is defined as

$$\text{GK.dim}A = \inf \{ \lambda \in \mathbb{R} \mid d_F(m) \leq m^\lambda \text{ for } m \gg 0 \}.$$

It is known that  $\text{GK.dim}A$  does not depend on the choice of a finite dimensional generating subspace of  $A$ . So, by the definition,  $\text{GK.dim}A$  amounts to a measure of the growth rate of  $A$  as a  $K$ -algebra with respect to any finite dimensional generating subspace. If the “inf” exists in the above definition, say  $\text{GK.dim}A = \lambda$ , then we say that  $A$  has polynomial growth; if the “inf” does not exist, then we write  $\text{GK.dim}A = \infty$ .

By means of a Gröbner basis technique, the next proposition tells us when a finitely generated  $K$ -algebra  $A = K[a_1, \dots, a_n]$  of  $n$  generators has  $\text{GK.dim}A = n$ .

**1.2. Proposition** [Li2, Section 6, Example 1] (or [Li4, Ch. 5, Section 5.3, Example 3]) Let the  $K$ -algebra  $A = K[a_1, \dots, a_n]$  be presented as a quotient algebra of the free  $K$ -algebra  $K\langle X \rangle = K\langle X_1, \dots, X_n \rangle$ , say  $A = K\langle X \rangle / I$ , and suppose that the ideal  $I$  of  $K\langle X \rangle$  has a finite Gröbner basis  $\mathcal{G} = \{g_{ji} \mid 1 \leq i < j \leq n\}$  with respect to some monomial ordering  $\prec_x$  on the  $K$ -basis  $\mathbb{B}$  of  $K\langle X \rangle$ , such that the leading monomial  $\mathbf{LM}(g_{ji}) = X_j X_i$ ,  $1 \leq i < j \leq n$ . Then  $A$  has polynomial growth and  $\text{GK.dim}A = n =$  the number of generators of  $A$ . □

Finally we recall the notion of Gelfand-Kirillov dimension (GK dimension for short) for finitely generated modules over a finitely generated  $K$ -algebra  $A = K[a_1, \dots, a_n]$ . Let  $M = \sum_{i=1}^s A\xi_i$  be a finitely generated left  $A$ -module with the generating set  $\{\xi_1, \dots, \xi_s\}$ . Considering  $A$  as an  $\mathbb{N}$ -filtered algebra with the standard filtration  $FA = \{F_mA\}_{m \in \mathbb{N}}$ ,  $M$  is then turned into an  $\mathbb{N}$ -filtered  $A$ -module by the filtration  $FM = \{F_q M\}_{q \in \mathbb{N}}$  with

$$F_q M = F_q A F_0 M, \text{ where } F_0 M = \sum_{i=1}^s K\xi_i, \quad q \in \mathbb{N},$$

which satisfies that each  $F_q M$  is a finite dimensional subspace of  $M$ ,  $M = \bigcup_{q \in \mathbb{N}} F_q M$ ,  $F_q M \subseteq F_{q+1} M$ ,  $F_m A F_q M \subseteq F_{m+q} M$  for all  $m, q \in \mathbb{N}$ . In the literature (cf. [MR,

P.300]), the filtration  $FM$  as described above is also referred to as the *standard filtration* of  $M$  determined by the *finite dimensional generating subspace*  $\sum_{i=1}^s K\xi_i$ . Considering the function  $d_F(q) = \dim_K F_q M$ ,  $q \in \mathbb{N}$ , the GK dimension of  $M$ , denoted  $\text{GK.dim} M$ , is defined as

$$\text{GK.dim} M = \inf \{ \lambda \in \mathbb{R} \mid d_F(q) \leq q^\lambda \text{ for } q \gg 0 \}.$$

It is also known that  $\text{GK.dim} M$  does not depend on the choice of a finite dimensional generating subspace of  $M$ . So, by the definition,  $\text{GK.dim} M$  amounts to a measure of the growth rate of  $M$  as an  $A$ -module with respect to any finite dimensional generating subspace. If the “inf” exists in the above definition, say  $\text{GK.dim} M = \lambda$ , then we say that  $M$  has polynomial growth; if the “inf” does not exist, then we write  $\text{GK.dim} M = \infty$ .

## 2. Elimination Lemma for Algebras with PBW Bases

Let  $A = K[a_1, \dots, a_n]$  be a finitely generated  $K$ -algebra with the PBW  $K$ -basis  $\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$ . Consider the positive-degree function  $d$  on  $\mathcal{B}$ , which assigns  $d(a_i) = 1$ ,  $1 \leq i \leq n$ . Then, for each  $a^\alpha \in \mathcal{B}$  with  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ , we have  $d(a^\alpha) = \alpha_1 + \cdots + \alpha_n$ . Thus the  $K$ -vector space  $A$  has the  $\mathbb{N}$ -filtration  $\mathcal{F}A = \{\mathcal{F}_m A\}_{m \in \mathbb{N}}$  with

$$\mathcal{F}_m A = K\text{-span} \{a^\alpha \in \mathcal{B} \mid d(a^\alpha) \leq m\}, \quad m \in \mathbb{N},$$

which satisfies that  $\mathcal{F}_0 A = K$ , each  $\mathcal{F}_m A$  is a finite dimensional subspace of  $A$ ,  $A = \cup_{m \in \mathbb{N}} \mathcal{F}_m A$ , and  $\mathcal{F}_m A \subseteq \mathcal{F}_{m+1} A$  for all  $m \in \mathbb{N}$ , but does not necessarily satisfy  $\mathcal{F}_{m_1} A \mathcal{F}_{m_2} A \subseteq \mathcal{F}_{m_1+m_2} A$  for  $m_1, m_2 \in \mathbb{N}$ , or in other words,  $A$  is not necessarily an  $\mathbb{N}$ -filtered algebra with respect to the filtration  $\mathcal{F}A$  (see the example given in Section 4). Furthermore, comparing the filtration  $\mathcal{F}A = \{\mathcal{F}_m A\}_{m \in \mathbb{N}}$  with the standard filtration  $FA = \{F_m A\}_{m \in \mathbb{N}}$  of  $A$  defined in the last section, it is clear that  $\mathcal{F}_m A \subseteq F_m A$ ,  $m \in \mathbb{N}$ .

Now, for any subset  $U_r = \{a_{i_1}, \dots, a_{i_r}\} \subset \{a_1, \dots, a_n\}$  with  $i_1 < i_2 < \cdots < i_r$ , let us take

$$\mathbf{V}(U_r) = K\text{-span} \left\{ a_{i_1}^{\alpha_1} \cdots a_{i_r}^{\alpha_r} \mid (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r \right\}.$$

Then we are ready to establish an Elimination Lemma for the algebra  $A$ .

**2.1. Elimination Lemma** Let  $A$  and the notations be as fixed above, and let  $L$  be a nonzero left ideal of  $A$ . If the left  $A$ -module  $A/L$  has finite GK dimension  $\text{GK.dim}(A/L) = d$ , then, for any subset  $U_r = \{a_{i_1}, \dots, a_{i_r}\} \subset \{a_1, \dots, a_n\}$  with  $i_1 < i_2 < \cdots < i_r$ ,  $\mathbf{V}(U_r) \cap L = \{0\}$  implies  $r \leq d$ . Consequently, if  $d < n$  (= the number of generators of  $A$ ), then  $\mathbf{V}(U_{d+1}) \cap L \neq \{0\}$  holds true for every subset  $U_{d+1} = \{a_{i_1}, \dots, a_{i_{d+1}}\} \subset \{a_1, \dots, a_n\}$



with  $i_1 < i_2 < \dots < i_{d+1}$ , in particular, for every  $U_s = \{a_1, \dots, a_s\}$  with  $d+1 \leq s \leq n-1$  we have  $\mathbf{V}(U_s) \cap L \neq \{0\}$ .

**Proof** Let the  $A$ -module  $A/L$  be equipped with the filtration  $F(A/L) = \{F_q(A/L)\}_{q \in \mathbb{N}}$  induced by the standard filtration  $FA = \{F_q A\}_{q \in \mathbb{N}}$  of  $A$ , i.e.,  $F_q(A/L) = (F_q A + L)/L$ ,  $q \in \mathbb{N}$ , which is clearly the standard filtration of  $A/L$  as defined in Section 1. Taking any  $U_r = \{a_{i_1}, \dots, a_{i_r}\} \subset \{a_1, \dots, a_n\}$  with  $i_1 < i_2 < \dots < i_r$ , consider the filtration  $\mathcal{F}\mathbf{V}(U_r) = \{\mathcal{F}_q \mathbf{V}(U_r)\}_{q \in \mathbb{N}}$  of the vector space  $\mathbf{V}(U_r)$  induced by the filtration  $\mathcal{F}A = \{\mathcal{F}_q A\}_{q \in \mathbb{N}}$  of  $A$  determined by the PBW basis  $\mathcal{B}$  (as defined above), i.e.,  $\mathcal{F}_q \mathbf{V}(U_r) = \mathbf{V}(U_r) \cap \mathcal{F}_q A$ ,  $q \in \mathbb{N}$ . If  $\mathbf{V}(U_r) \cap L = \{0\}$ , then since

$$\mathcal{F}_q \mathbf{V}(U_r) \cong \frac{\mathcal{F}_q \mathbf{V}(U_r) + L}{L} \subset \frac{\mathcal{F}_q A + L}{L} \subset \frac{F_q A + L}{L}, \quad q \in \mathbb{N},$$

it turns out that for every  $q \in \mathbb{N}$ ,

$$\begin{aligned} \binom{q+r}{r} &= \dim_K \mathcal{F}_q \mathbf{V}(U_r) \\ &= \dim_K \frac{\mathcal{F}_q \mathbf{V}(U_r) + L}{L} \\ &\leq \dim_K \frac{F_q A + L}{L} \\ &= \dim_K F_q(A/L). \end{aligned}$$

Now if  $\text{GK.dim} M = d$ , then, taking the usual infimum

$$\inf\{\lambda \in \mathbb{R} \mid \dim_K \mathcal{F}_q \mathbf{V}(U_r) \leq q^\lambda, \quad q \gg 0\}$$

into account (of course this infimum is nothing about GK dimension), it follows from the definition of GK dimension for the module  $M$  that

$$\begin{aligned} r &= \inf\{\lambda \in \mathbb{R} \mid \dim_K \mathcal{F}_q \mathbf{V}(U_r) \leq q^\lambda, \quad q \gg 0\} \\ &\leq \inf\{\lambda \in \mathbb{R} \mid \dim_K F_q(A/L) \leq q^\lambda, \quad q \gg 0\} \\ &= \text{GK.dim}(A/L) = d, \end{aligned}$$

as desired. Consequently, the last assertion of the lemma is immediately clear.  $\square$

For convenience in using Elimination Lemma 2.1, it is reasonable to introduce

**2.2. Definition** Let  $A = K[a_1, \dots, a_n]$  be a finitely generated  $K$ -algebra with the PBW  $K$ -basis, and let  $L$  be a nonzero left ideal of  $A$ . If

$$(*) \quad \text{GK.dim}(A/L) < n$$

then we say that *Elimination Lemma 2.1 holds true for  $L$* .

Since by [Li1, CH.V, Theorem 7.4] we know that every nonzero left ideal  $L$  of a quadric solvable polynomial algebra  $A = K[a_1, \dots, a_n]$  satisfies  $\text{GK.dim}(A/L) < \text{GK.dim}A = n$ , it follows that  $L$  satisfies the condition  $(*)$  of Definition 2.2, and hence Elimination Lemma 2.1 holds true for every nonzero left ideal  $L$  of  $A$ . Thereby Elimination Lemma 2.1 covers the elimination lemma for quadric solvable polynomial algebras [Li1, Lemma 7.5], especially it covers the elimination lemma for Weyl algebras [Zei, Lemma 4.1] (note that Weyl algebras are typical quadric solvable polynomial algebras).

Let  $A$  be an arbitrary finitely generated  $K$ -algebra with the PBW  $K$ -basis  $\mathcal{B}$ , and  $L$  a nonzero left ideal of  $A$ . From a practical viewpoint, it seems that computing  $\text{GK.dim}(A/L)$  in an algorithmic way (as in [Li1, CH.V]) is not always feasible in order to realize the condition  $(*)$  of Definition 2.2 for  $L$ . Instead, learning from the knowledge of Gelfand-Kirillov dimension for algebras and modules, we would rather try to determine whether

$$\text{GK.dim}(A/L) < \text{GK.dim}A \leq n \text{ (= the number of generators of } A).$$

As to getting the first inequality  $\text{GK.dim}(A/L) < \text{GK.dim}A$ , the lemma presented below may shed light on this topic.

**2.3. Lemma** Let  $A$  be any  $K$ -algebra. Then the following statements hold.

- (i) If  $f \in A$  is not a divisor of zero, then  $\text{GK.dim}(A/Af) \leq \text{GK.dim}A - 1$ .
- (ii) If  $A$  is a domain and  $L$  is any nonzero left ideal of  $A$ , then  $\text{GK.dim}(A/L) \leq \text{GK.dim}A - 1$ .

**Proof** (i) If  $f \in A$  is not a divisor of zero, then  $A \cong Af$  as left  $A$ -modules. It follows from [MR, Proposition 8.3.5] that  $\text{GK.dim}(A/Af) \leq \text{GK.dim}A - 1$ .

(ii) If  $A$  is a domain and  $L$  is any nonzero left ideal of  $A$ , then, taking a nonzero  $f \in L$ , It follows from (i) and the exact sequence  $A/Af \rightarrow A/L \rightarrow 0$  of  $A$ -modules that  $\text{GK.dim}(A/L) \leq \text{GK.dim}(A/Af) \leq \text{GK.dim}A - 1$ .  $\square$

Concerning the second inequality  $\text{GK.dim}A \leq n$  (= the number of generators of  $A$ ), it is certainly a matter of determining  $\text{GK.dim}A$ . As one may know from the literature, there are many different ways to determine the GK dimension of an algebra. Especially for a finitely generated  $K$ -algebra  $A = K[a_1, \dots, a_n]$ , if  $A$  is presented as a quotient algebra  $K\langle X \rangle / I$  of the free  $K$ -algebra  $K\langle X \rangle = K\langle X_1, \dots, X_n \rangle$  and if the ideal  $I$  has a finite Gröbner basis  $\mathcal{G}$  with respect to some monomial ordering  $\prec$ , then it follows from [Uf] that  $\text{GK.dim}A$  can be read out from the Ufnarovski graph of  $\mathcal{G}$  (see also [Li2, Section 6] and

[Li4, Ch.5] for a number of examples including the foregoing Proposition 1.2). While for an arbitrary finitely generated  $K$ -algebra  $A$  with the PBW  $K$ -basis, the next lemma may also help us to determine the GK dimension of  $A$ .

**2.4. Lemma** Let  $A = K[a_1, \dots, a_n]$  be a finitely generated  $K$ -algebra with the PBW  $K$ -basis  $\mathcal{B}$ , and let  $\mathcal{F}A = \{\mathcal{F}_m A\}_{m \in \mathbb{N}}$  be the  $\mathbb{N}$ -filtration of the vector space  $A$  determined by  $\mathcal{B}$  as described before, i.e.,  $\mathcal{F}_m A = K\text{-span}\{a^\alpha \in \mathcal{B} \mid d(a^\alpha) \leq m\}$ ,  $m \in \mathbb{N}$ . Then, the following two statements are equivalent:

(i) The generators of  $A$  satisfy

$$a_j a_i = \sum_{q \leq \ell} \lambda_{q\ell} a_q a_\ell + \sum_t \lambda_t a_t + \lambda_{ji}, \quad 1 \leq i < j \leq n, \quad \lambda_{q\ell}, \lambda_t, \lambda_{ji} \in K.$$

(ii)  $A$  is turned into an  $\mathbb{N}$ -filtered algebra by  $\mathcal{F}A$ , i.e.,  $\mathcal{F}_{m_1} A \mathcal{F}_{m_2} A \subseteq \mathcal{F}_{m_1+m_2} A$  for all  $m_1, m_2 \in \mathbb{N}$ ,

**Proof** Note that the filtration  $\mathcal{F}A$  is constructed by using the positive-degree function  $d(\cdot)$  on  $\mathcal{B}$  such that  $d(a_i) = 1$ ,  $1 \leq i \leq n$ . So it is straightforward to verify that  $A$  is turned into an  $\mathbb{N}$ -filtered algebra by  $\mathcal{F}A$  if and only if  $a_j a_i$  has the desired representation.  $\square$

**2.5. Proposition** Let  $A = K[a_1, \dots, a_n]$  be a finitely generated  $K$ -algebra with the PBW  $K$ -basis  $\mathcal{B}$ . Consider the standard filtration  $FA = \{F_m A\}_{m \in \mathbb{N}}$  of the  $\mathbb{N}$ -filtered algebra  $A$ , and the  $\mathbb{N}$ -filtration  $\mathcal{F}A = \{\mathcal{F}_m A\}_{m \in \mathbb{N}}$  of the vector space  $A$  determined by the PBW  $K$ -basis  $\mathcal{B}$ , as described before. If  $A$  is also an  $\mathbb{N}$ -filtered algebra with respect to the filtration  $\mathcal{F}A$ , then  $F_m A = \mathcal{F}_m A$  for all  $m \in \mathbb{N}$ , and consequently  $\text{GK.dim} A = n$ .

**Proof** By the construction of both  $FA$  and  $\mathcal{F}A$ , it is clear that  $\mathcal{F}_m A \subseteq F_m A$  holds for all  $m \in \mathbb{N}$ . On the other hand, if  $A$  is also an  $\mathbb{N}$ -filtered algebra with respect to the filtration  $\mathcal{F}A$ , then applying Lemma 2.4 (i) to the structure of both filtration  $FA$  and  $\mathcal{F}A$ , the inclusion  $F_m A \subseteq \mathcal{F}_m A$  is obtained for every  $m \in \mathbb{N}$ . It follows that  $F_m A = \mathcal{F}_m A$  for all  $m \in \mathbb{N}$ , and consequently

$$\binom{m+n}{n} = \dim_K \mathcal{F}_m A = \dim_K F_m A, \quad m \in \mathbb{N}.$$

Thereby the knowledge of Gelfand-Kirillov dimension for algebras entails

$$\begin{aligned} n &= \inf\{\lambda \in \mathbb{R} \mid \dim_K \mathcal{F}_m A \leq m^\lambda, \quad m \gg 0\} \\ &= \inf\{\lambda \in \mathbb{R} \mid \dim_K F_m A \leq m^\lambda, \quad m \gg 0\} \\ &= \text{GK.dim} A, \end{aligned}$$

as desired. □

With the aid of the foregoing preparation, the next theorem provides us with a class of algebras such that if an algebra  $A$  belongs to this class, then every nonzero left ideal of  $A$  satisfies the condition  $(*)$  of Definition 2.2.

**2.6. Theorem** Let  $A = K[a_1, \dots, a_n]$  be a finitely generated  $K$ -algebra. If

- (1)  $A$  has the PBW  $K$ -basis  $\mathcal{B}$ ,
- (2) the generators of  $A$  satisfy

$$a_j a_i = \sum_{q \leq \ell} \lambda_{q\ell} a_q a_\ell + \sum_t \lambda_t a_t + \lambda_{ji}, \quad 1 \leq i < j \leq n, \quad \lambda_{q\ell}, \lambda_t, \lambda_{ji} \in K,$$

and

- (3)  $A$  is a domain,

then  $\text{GK.dim} A = n$ , and Elimination lemma 2.1 holds true (in the sense of Definition 2.2) for every nonzero left ideal  $L$  of  $A$ .

**Proof** This is just a result of combining Lemma 2.3, Lemma 2.4 and Proposition 2.5. □

**Remark** In the next two sections, we shall respectively determine two significant subclasses of the class of finitely generated  $K$ -algebras with the PBW  $K$ -basis  $\mathcal{B}$ , such that if an algebra  $A$  belongs to either of the two subclasses then Elimination Lemma holds true (in the sense of Definition 2.2) for every nonzero left ideal  $L$  of  $A$ ; moreover, the two subclasses of algebras will also illustrate that

- (i) if a finitely generated  $K$ -algebra  $A = K[a_1, \dots, a_n]$  is presented as a quotient algebra of the free  $K$ -algebra  $K\langle X \rangle = K\langle X_1, \dots, X_n \rangle$ , i.e.,  $A = K\langle X \rangle / I$ , then to a large extent, the Gröbner basis technique as shown in Proposition 1.1 will be quite helpful for us to check whether the conditions (1) and (2) of Theorem 2.6 are satisfied by  $A$ ;
- (ii) the class of algebras satisfying the three conditions of Theorem 2.6 properly contains the class of all quadric solvable polynomial algebras in the sense of [Li1, CH.III, Section 2];
- (iii) a finitely generated  $K$ -algebra  $A$  satisfying the conditions (1) and (3) of Theorem 2.6 and such that Elimination lemma 2.1 holds true (in the sense of Definition 2.2) for every nonzero left ideal, may not necessarily satisfy the condition (2) of Theorem 2.6 (see the example given in Section 4).

### 3. An Application to Binomial Skew Polynomial Rings

In the algebraic study of finding solutions to Yang-Baxter equations, the class of binomial skew polynomial rings was introduced and studied in [G-I1,2], and quite rich results were obtained in which the most important results are: every binomial skew-polynomial ring  $A$  is respectively

- (1) a left and right Noetherian domain;
- (2) an Artin-Schelter regular PBW algebra;
- (3) a Koszul algebra such that the Koszul dual  $A^!$  is a quantum Grassmann algebra;
- (4) a quantum binomial PBW algebra in the sense of [G-I3], and hence a Yang-Baxter algebra, that is, the set of defining relations  $\mathcal{R}$  of  $A$  defines canonically a solution to the Yang-Baxter equation.

In this section we demonstrate that every binomial skew-polynomial ring  $A$  satisfies the three conditions of Theorem 2.6. To better understand this result, with notions and notations as used in previous sections, we start by recalling from loc cit. the definition of a binomial skew-polynomial ring.

**3.1. Definition** Let  $K\langle X \rangle = \langle X_1, \dots, X_n \rangle$  be the free  $K$ -algebra generated by  $X = \{X_1, \dots, X_n\}$  with the standard  $K$ -basis  $\mathbb{B} = \{1\} \cup \{X_{i_1} \cdots X_{i_s} \mid X_{i_j} \in X, s \geq 1\}$ , and let  $I$  be an ideal of  $K\langle X \rangle$ ,  $A = K\langle X \rangle / I$ . If every  $X_i$  is assigned the degree 1, and if  $I$  is generated by the subset  $\mathcal{R} = \{R_{ji} = X_j X_i - c_{ij} X_{i'} X_{j'}\}_{1 \leq i < j \leq n}$  consisting of exactly  $\frac{n(n-1)}{2}$  elements, such that

- (a)  $c_{ij} \in K^* = K - \{0\}$ ,  $1 \leq i < j \leq n$ ;
- (b) each  $R_{ji} = X_j X_i - c_{ij} X_{i'} X_{j'}$  satisfies  $i' < j, i' < j', 1 \leq i < j \leq n$ ;
- (c)  $\{X_{i'} X_{j'} \mid X_j X_i - c_{ij} X_{i'} X_{j'} = R_{ji} \in \mathcal{R}\} = \{X_i X_j \mid 1 \leq i < j \leq n\}$ ;
- (d) with respect to the graded lexicographic monomial ordering  $X_1 \prec_{grlex} X_2 \prec_{grlex} \cdots \prec_{grlex} X_n$ ,  $\mathcal{R}$  forms a reduced Gröbner basis of  $I$ .

then  $A$  is called a *binomial skew polynomial ring*.

**3.2. Theorem** Let  $A = K\langle X \rangle / I$  be a binomial skew polynomial ring as defined above. Then  $A$  satisfies the three conditions of Theorem 2.6, thereby Elimination lemma 2.1 holds true (in the sense of Definition 2.2) for every nonzero left ideal  $L$  of  $A$ .

**Proof** Though this assertion may follow directly from the definition and the structural properties of a binomial skew polynomial ring as we listed in the beginning of this section, we give a step-by-step argument as follows. By the condition (d) of Definition 3.1, Proposition 1.1 entails that  $A$  has the PBW  $K$ -basis, i.e. the condition (1) of Theorem 2.6 is satisfied. By the definition of  $\mathcal{R}$ ,  $A$  satisfies the condition (2) of Theorem 2.6. Since

every binomial skew polynomial ring is a domain as established in [G-I1,2], the condition (3) of Theorem 2.6 is satisfied by  $A$  as well.  $\square$

Let  $A = K\langle X \rangle / I = K[a_1, \dots, a_n]$  be a binomial skew polynomial ring as defined above, where  $a_i$  stands for the coset  $X_i + I$  in  $K\langle X \rangle / I$ ,  $1 \leq i \leq n$ . Then  $A$  is clearly a *skew 2-nomial algebra* in the sense of [Li3], and it follows from the conditions (a) – (d) of Definition 3.1 that numerous  $A$  may fall into the case of [Li3, Theorem 2.2]. If it is the case, then  $A$  has the lexicographic ordering  $\prec_{lex}$  on the PBW  $K$ -basis  $\mathcal{B}$  such that  $a_n \prec_{lex} a_{n-1} \prec_{lex} \dots \prec_{lex} a_2 \prec_{lex} a_1$ , and  $A$  has a left Gröbner basis theory with respect to  $\prec_{lex}$ , i.e., every left ideal  $L$  of  $A$  has a finite left Gröbner basis in the sense that if  $f \in L$  and  $f \neq 0$ , then there is a  $g \in \mathcal{G}$  such that  $\mathbf{LM}(g) | \mathbf{LM}(f)$  (a left division), where  $\mathbf{LM}(\ )$  stands for taking the leading monomial of elements in  $A$  with respect to  $\prec_{lex}$  (note that a binomial skew polynomial ring  $A$  is Noetherian and hence a left Gröbner basis of  $L$  is finite). Thus, since for each  $s = 1, 2, \dots, n-1$  the monomial ordering  $\prec_{lex}$  is an elimination ordering of type  $s$  with respect to  $\mathbf{V}(U_s)$ , i.e.,  $f \in A$  with the leading monomial  $\mathbf{LM}(f) \in \mathbf{V}(U_s)$  implies  $f \in \mathbf{V}(U_s)$ , where  $U_s = \{a_n, \dots, a_{n-s+1}\}$ ,  $\mathbf{V}(U_s) = K\text{-span}\{a_{n-s+1}^{\alpha_1} \dots a_n^{\alpha_s} \mid (\alpha_1, \dots, \alpha_s) \in \mathbb{N}^s\}$ , if  $L$  is a nonzero left ideal of  $A$ , then  $\text{GK.dim}(A/L) = d < \text{GK.dim} A = n$ ,  $\mathbf{V}(U_{d+1}) \cap L \neq \{0\}$  by Theorem 3.2 and Lemma 2.1, and a left Gröbner basis  $\mathcal{G}$  of  $L$  with respect to  $\prec_{lex}$  will contain a nonzero element of  $\mathbf{V}(U_{d+1})$ . Therefore, realizing the elimination property of  $L$  (as described in Lemma 2.1) may become possible in case computing a left Gröbner basis for  $L$  under  $\prec_{lex}$  is algorithmically feasible.

By Definition 3.1 it is also clear that a binomial skew polynomial ring  $A$  is not necessarily a quadric solvable polynomial algebra in the sense of [Li1, CH.III, Section 2]. Hence, with Theorem 3.2 we end this section by concluding that the class of algebras satisfying the three conditions of Theorem 2.6 properly contains the class of all quadric solvable polynomial algebras in the sense of [Li1, CH.III, Section 2]. This illustrates Remark (ii) given at the end of Section 2.

## 4. An Application to Solvable Polynomial Algebras

As we have seen from the introduction section and Section 2, that Elimination Lemma 2.1 holds true for every nonzero left ideal of a quadric solvable polynomial algebra in the sense of [Li1, CH.III, Section 2]. In this section, we show that, indeed, Elimination Lemma 2.1 holds true (in the sense of Definition 2.2) for every nonzero left ideal of an arbitrary solvable polynomial algebra in the sense of [K-RW]. To this end, we first recall

the following

**4.1. Definition** ([K-RW], [LW]) Let  $A = K[a_1, \dots, a_n]$  be a finitely generated  $K$ -algebra. Suppose that  $A$  has the PBW  $K$ -basis  $\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$ , and that  $\prec$  is a (two-sided) monomial ordering on  $\mathcal{B}$ . If for all  $a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n}$ ,  $a^\beta = a_1^{\beta_1} \cdots a_n^{\beta_n} \in \mathcal{B}$ , the following holds:

$$\begin{aligned} a^\alpha a^\beta &= \lambda_{\alpha,\beta} a^{\alpha+\beta} + f_{\alpha,\beta}, \\ \text{where } \lambda_{\alpha,\beta} &\in K^*, \quad a^{\alpha+\beta} = a_1^{\alpha_1+\beta_1} \cdots a_n^{\alpha_n+\beta_n}, \text{ and either } f_{\alpha,\beta} = 0 \text{ or} \\ f_{\alpha,\beta} &= \sum_q \mu_q a^{\gamma(q)} \in A \text{ with } \mu_q \in K, \quad a^{\gamma(q)} \in \mathcal{B}, \text{ satisfying } \mathbf{LM}(f_{\alpha,\beta}) \prec a^{\alpha+\beta}, \end{aligned}$$

where  $\mathbf{LM}(f_{\alpha,\beta})$  stands for the leading monomial of  $f_{\alpha,\beta}$  with respect to  $\prec$ , then  $A$  is called a *solvable polynomial algebra*.

**4.2. Proposition** [Li5, Theorem 2.1] Let  $A = K[a_1, \dots, a_n]$  be a finitely generated  $K$ -algebra, and let  $K\langle X \rangle = K\langle X_1, \dots, X_n \rangle$  be the free  $K$ -algebras with the standard  $K$ -basis  $\mathbb{B} = \{1\} \cup \{X_{i_1} \cdots X_{i_s} \mid X_{i_j} \in X, s \geq 1\}$ . With notations as before, the following two statements are equivalent:

- (i)  $A$  is a solvable polynomial algebra in the sense of Definition 4.1.
- (ii)  $A \cong \overline{A} = K\langle X \rangle / I$  via the  $K$ -algebra epimorphism  $\pi_1: K\langle X \rangle \rightarrow A$  with  $\pi_1(X_i) = a_i$ ,  $1 \leq i \leq n$ ,  $I = \text{Ker} \pi_1$ , satisfying
  - (a) with respect to some monomial ordering  $\prec_x$  on  $\mathbb{B}$ , the ideal  $I$  has a finite Gröbner basis  $G$  and the reduced Gröbner basis of  $I$  is of the form

$$\mathcal{G} = \left\{ g_{ji} = X_j X_i - \lambda_{ji} X_i X_j - F_{ji} \mid \begin{array}{l} \mathbf{LM}(g_{ji}) = X_j X_i, \\ 1 \leq i < j \leq n \end{array} \right\}$$

where  $\lambda_{ji} \in K^*$ ,  $\mu_q^{ji} \in K$ , and  $F_{ji} = \sum_q \mu_q^{ji} X_1^{\alpha_{1q}} X_2^{\alpha_{2q}} \cdots X_n^{\alpha_{nq}}$  with  $(\alpha_{1q}, \alpha_{2q}, \dots, \alpha_{nq}) \in \mathbb{N}^n$ , thereby  $\mathcal{B} = \{\overline{X}_1^{\alpha_1} \overline{X}_2^{\alpha_2} \cdots \overline{X}_n^{\alpha_n} \mid \alpha_j \in \mathbb{N}\}$  forms a PBW  $K$ -basis for  $\overline{A}$ , where each  $\overline{X}_i$  denotes the coset of  $I$  represented by  $X_i$  in  $\overline{A}$ ; and

- (b) there is a (two-sided) monomial ordering  $\prec$  on  $\mathcal{B}$  such that  $\mathbf{LM}(\overline{F}_{ji}) \prec \overline{X}_i \overline{X}_j$  whenever  $\overline{F}_{ji} \neq 0$ , where  $\overline{F}_{ji} = \sum_q \mu_q^{ji} \overline{X}_1^{\alpha_{1q}} \overline{X}_2^{\alpha_{2q}} \cdots \overline{X}_n^{\alpha_{nq}}$ ,  $1 \leq i < j \leq n$ .

□

In conclusion, we derive the next

**4.3. Theorem** Let  $A = K[a_1, \dots, a_n]$  be any solvable polynomial algebra in the sense of Definition 4.1. Then Elimination Lemma 2.1 holds true (in the sense of Definition 2.2) for every nonzero left ideal  $L$  of  $A$ .

**Proof** First note that every solvable polynomial algebra  $A$  has the PBW  $K$ -basis  $\mathcal{B}$  by Definition 4.1 (or by Proposition 1.1 and Theorem 4.2). Moreover, it follows from Proposition 1.2 and Proposition 4.2 that  $\text{GK.dim} A = n$  (= the number of generators of  $A$ ). As also we know that  $A$  is a domain by [K-RW]. Hence Lemma 2.3 entails that  $\text{GK.dim}(A/L) < \text{GK.dim} A = n$  holds for every nonzero left ideal of  $A$ . Therefore, we conclude that Elimination Lemma 2.1 holds true (in the sense of Definition 2.2) for every nonzero left ideal  $L$  of  $A$ .  $\square$

Comparing with Theorem 2.6, we see that Theorem 4.3 did not require  $A$  satisfies the condition (2) of Theorem 2.6 (or equivalently, Theorem 4.3 did not require  $A$  is an  $\mathbb{N}$ -filtered algebra with respect to the filtration  $\mathcal{F}A$  determined by the PBW  $K$ -basis  $\mathcal{B}$ ). Indeed, the example presented below illustrates that a solvable polynomial algebra  $A$  may not necessarily satisfy the condition (2) of Theorem 2.6 (or equivalently,  $A$  may not be an  $\mathbb{N}$ -filtered algebra with respect to the filtration  $\mathcal{F}A$  determined by the PBW  $K$ -basis  $\mathcal{B}$ ).

**Example** [Li5, Example 1] Considering the positive-degree function  $d$  on the free  $K$ -algebra  $K\langle X \rangle = K\langle X_1, X_2, X_3 \rangle$  such that  $d(X_1) = 2$ ,  $d(X_2) = 1$ , and  $d(X_3) = 4$ , let  $I$  be the ideal of  $K\langle X \rangle$  generated by the subset  $\mathcal{G}$  consisting of

$$\begin{aligned} g_1 &= X_1X_2 - X_2X_1, \\ g_2 &= X_3X_1 - \lambda X_1X_3 - \mu X_3X_2^2 - f(X_2), \\ g_3 &= X_3X_2 - X_2X_3, \end{aligned}$$

where  $\lambda \in K^*$ ,  $\mu \in K$ ,  $f(X_2)$  is a polynomial in  $X_2$  with  $d(f(X_2)) \leq 6$ , or  $f(X_2) = 0$ . The following statements hold.

- (1)  $\mathcal{G}$  forms a Gröbner basis for  $I$  with respect to the graded lexicographic ordering  $X_2 \prec_{\text{grlex}} X_1 \prec_{\text{grlex}} X_3$ , such that the three generators of  $I$  have the leading monomials  $\mathbf{LM}(g_1) = X_1X_2$ ,  $\mathbf{LM}(g_2) = X_3X_1$ , and  $\mathbf{LM}(g_3) = X_3X_2$ .
- (2) With respect to the fixed  $\prec_{\text{grlex}}$  in (1), the reduced Gröbner basis  $\mathcal{G}'$  of  $I$  consists of

$$\begin{aligned} g_1 &= X_1X_2 - X_2X_1, \\ g_2 &= X_3X_1 - \lambda X_1X_3 - \mu X_2^2X_3 - f(X_2), \\ g_3 &= X_3X_2 - X_2X_3, \end{aligned}$$

- (3) Writing  $A = K[a_1, a_2, a_3]$  for the quotient algebra  $K\langle X \rangle/I$ , where  $a_1$ ,  $a_2$  and  $a_3$  denote the cosets  $X_1 + I$ ,  $X_2 + I$  and  $X_3 + I$  in  $K\langle X \rangle/I$  respectively, it follows that  $A$  has the PBW basis  $\mathcal{B} = \{a^\alpha = a_2^{\alpha_2} a_1^{\alpha_1} a_3^{\alpha_3} \mid \alpha = (\alpha_2, \alpha_1, \alpha_3) \in \mathbb{N}^3\}$ . Noticing that  $a_2a_1 = a_1a_2$ , it is clear that  $\mathcal{B}' = \{a^\alpha = a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_3} \mid \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3\}$  is also a PBW basis for  $A$ . Since  $a_3a_1 = \lambda a_1a_3 + \mu a_2^2a_3 + f(a_2)$ , where  $f(a_2) \in K\text{-span}\{1, a_2, a_2^2, \dots, a_2^6\}$ , we see that  $A$  has



the monomial ordering  $\prec_{lex}$  on  $\mathcal{B}'$  such that  $a_3 \prec_{lex} a_2 \prec_{lex} a_1$  and  $\mathbf{LM}(\mu a_2^2 a_3 + f(a_2)) \prec_{lex} a_1 a_3$ , thereby  $A$  is turned into a solvable polynomial algebra with respect to  $\prec_{lex}$ . Also if we use the positive-degree function  $d$  on  $\mathcal{B}'$  such that  $d(a_1) = 2$ ,  $d(a_2) = 1$ , and  $d(a_3) = 4$ , then  $A$  has another monomial ordering on  $\mathcal{B}'$ , namely the graded lexicographic ordering  $\prec_{grlex}$  such that  $a_3 \prec_{grlex} a_2 \prec_{grlex} a_1$  and  $\mathbf{LM}(\mu a_2^2 a_3 + f(a_2)) \prec_{grlex} a_1 a_3$ , thereby  $A$  is turned into a solvable polynomial algebra with respect to  $\prec_{grlex}$ .

(4) Consider the  $K$ -algebra  $A = K[a_1, a_2, a_3]$  as presented in the above (3), and consider the positive-degree function  $d$  on the PBW  $K$ -basis  $\mathcal{B}'$  of  $A$  such that  $d(a_i) = 1$ ,  $1 \leq i \leq 3$ . Then  $A$  is not an  $\mathbb{N}$ -filtered algebra with respect to the filtration  $\mathcal{F}A$  determined by  $\mathcal{B}'$ , because  $a_3 a_1 = \lambda a_1 a_3 + \mu a_2^2 a_3 + f(a_2)$  implies  $\mathcal{F}_1 A \mathcal{F}_1 A \not\subset \mathcal{F}_2 A$ . Therefore, the solvable polynomial algebra  $A$  does not satisfies the condition (2) of Theorem 2.6 (see also Lemma 2.4), illustrating Remark (iii) given at the end of Section 2.

Finally, note that a noncommutative Gröbner basis theory works effectively for every solvable polynomial algebra  $A = K[a_1, \dots, a_n]$  in the sense of [K-RW], that is, a noncommutative Buchberger's algorithm works very well in the sense that if a finite generating set of a left ideal  $L$  is given (note that  $A$  is Noetherian), then running the noncommutative Buchberger's algorithm with respect to a monomial ordering  $\prec$  will produce a finite left Gröbner basis  $\mathcal{G}$  for  $L$ . Thus, since  $\text{GK.dim} A/L = d < n$  by Theorem 4.3, if  $U = \{a_{i_1}, \dots, a_{i_{d+1}}\}$  with  $i_1 < i_2 < \dots < i_{d+1}$ , and if  $A$  admits an elimination ordering  $\prec_{d+1}$  of type  $d+1$  with respect to  $\mathbf{V}(U) = K\text{-span}\{a_{i_1}^{\alpha_1} \cdots a_{i_{d+1}}^{\alpha_{d+1}} \mid (\alpha_1, \dots, \alpha_{d+1}) \in \mathbb{N}^{d+1}\}$ , i.e.,  $f \in A$  with the leading monomial  $\mathbf{LM}(f) \in \mathbf{V}(U)$  implies  $f \in \mathbf{V}(U)$ , then the noncommutative Buchberger's algorithm will produce a left Gröbner basis  $\mathcal{G}$  of  $L$  such that  $\mathcal{G}$  contains a nonzero element of  $\mathbf{V}(U)$ . This shows that the elimination property of  $L$  (as described in Lemma 2.1) may be realized via computing a left Gröbner basis of  $L$  with respect to a suitable elimination ordering.

## References

- [BW] T. Becker and V. Weispfenning, *Gröbner Bases*, Springer-Verlag, 1993.
- [Ch] F. Chyzak, Holonomic systems and automatic proving of identities, *Research Report* 2371, Institute National de Recherche en Informatique et en Automatique, 1994.
- [CS] F. Chyzak and B. Salvy, Noncommutative elimination in Ore algebras proves multivariate identities, *J. Symbolic Comput.*, 26(1998), 187–227.
- [G-I1] T. Gateva-Ivanova, Skew polynomial rings with binomial relations. *J. Algebra*, 185(1996), 710–753.
- [G-I2] T. Gateva-Ivanova, Binomial skew polynomial rings, Artin-Schelter regularity, and binomial solutions of the Yang-Baxter equation. *Serdica Math. J.* 30(2004), 431–470

- [G-I3] T. Gateva-Ivanova, Quadratic algebras, Yang-Baxter equation, and Artin-Schelter regularity *Advances in Mathematics*, 230(2012), 2152C2175.
- [Gr] E. L. Green, Noncommutative Gröbner bases and projective resolutions. In: *Computational Methods for Representations of Groups and Algebras* (Michler, Schneider, eds), Proceedings of the Euroconference, Essen, 1997. Progress in Mathematics, Vol. 173, Basel, Birkhauser Verlag, 1999, 29–60.
- [Grö] W. Gröbner, *Algebraic Geometrie* I, II, Bibliographisches Institut, Mannheim, 1968, 1970.
- [Hu] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*. Springer, 1972.
- [KL] G.R. Krause and T.H. Lenagan, *Growth of Algebras and Gelfand-Kirillov Dimension*. Graduate Studies in Mathematics. American Mathematical Society, 1991.
- [KW] H. Kredel and V. Weispfenning, Computing dimension and independent sets for polynomial ideals. In: *Computational Aspects of Commutative Algebra*, from a special issue of the Journal of Symbolic Computation (L. Robbiano ed), Academic Press, 1989, 97–113.
- [KR-W] A. Kandri-Rody, V. Weispfenning, Non-commutative Gröbner bases in algebras of solvable type. *J. Symbolic Comput.*, 9(1990), 1–26.
- [Li1] H. Li, *Noncommutative Gröbner Bases and Filtered-Graded Transfer*. Lecture Notes in Mathematics, Vol. 1795, Springer, 2002.
- [Li2] H. Li,  $\Gamma$ -leading homogeneous algebras and Gröbner bases. In: *Recent Developments in Algebra and Related Areas* (F. Li and C. Dong eds.), Advanced Lectures in Mathematics, Vol. 8, International Press & Higher Education Press, Boston-Beijing, 2009, 155–200. (This is a strengthened version of [arXiv:math.RA/0609583](https://arxiv.org/abs/math.RA/0609583), <http://arXiv.org>)
- [Li3] H. Li, Looking for Gröbner basis theory for (almost) skew 2-nomial algebras. *J. Symbolic Comput.*, 45(2010), 918C942. Available at [arXiv:0808.1477](https://arxiv.org/abs/0808.1477) [math.RA], <http://arXiv.org>
- [Li4] H. Li, *Gröbner Bases in Ring Theory*. World Scientific Publishing Company, 2011.
- [Li5] H. Li, A Note on solvable polynomial algebras. *Computer Science Journal of Moldova*, vol.22, 1(64), 2014, 99 – 109. Available at [arXiv:1212.5988](https://arxiv.org/abs/1212.5988) [math.RA], <http://arXiv.org>
- [LW] H. Li. Y. Wu, Filtered-graded transfer of Gröbner basis computation in solvable polynomial algebras. *Communications in Algebra*, 28(1), 2000, 15–32.
- [MR] J.C. McConnell and J.C. Robson, *Noncommutative Noetherian Rings*, John Wiley & Sons, 1987.

- [Mor] T. Mora, An introduction to commutative and noncommutative Gröbner Bases. *Theoretic Computer Science*, 134, 1994, 131–173.
- [PWZ] M. Petkovsek, H. Wilf and D. Zeilberger,  $A = B$ . A.K. Peters, Ltd. 1996.
- [SW] A.V. Shepler and S. Witherspoon, Poincaré-Birkhoff-Witt Theorems. In: *Commutative Algebra and Noncommutative Algebraic Geometry* (D. Eisenbud, S.B. Iyengar, A.K. Singh, J.T. Stafford, and M. Van den Bergh, eds), Mathematical Sciences Research Institute Proceedings, Vol. 1, Cambridge Univ. Press, 2015.
- [Uf] V. Ufnarovski, A growth criterion for graphs and algebras defined by words. *Mat. Zametki*, 31(1982), 465–472 (in Russian); English translation: *Math. Notes*, 37(1982), 238–241.
- [WZ] H.S. Wilf and D. Zeilberger, An algorithmic proof theory for hypergeometric (ordinary and q) multisum/integral identities, *Invent. Math.*, 108(1992), 575 – 633.
- [Zei1] D. Zeilberger, A holonomic system approach to special function identities, *J. Comput. Appl. Math.*, 32(1990), 321 – 368.
- [Zei2] D. Zeilberger, The method of creative telescoping, *J. Symbolic Comput.*, 11(1991), 195C204.